

Subsolutions of the Dirac equation and WIMP dark matter

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Abstract

In the present paper we study subsolutions of the Dirac equation. The known examples of such subsolutions are the Weyl neutrino and the Majorana neutrino. It is shown that the Dirac equation for a massive non-interacting particle admits another decomposition into two separate equations. It is suggested that such particles can be identified with weakly interacting massive particles, other than neutrinos and of non-exotic nature, which can be candidates for dark matter.

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I. INTRODUCTION

The Dirac equation is a relativistic quantum mechanical wave equation formulated by Paul Dirac in 1928 providing a description of elementary spin- $\frac{1}{2}$ particles, such as electrons and quarks, consistent with both the principles of quantum mechanics and the theory of special relativity [1]. The Dirac equation is [2, 3, 4]:

$$\gamma^\mu p_\mu \Psi = m \Psi, \quad (1)$$

where m is the rest mass of the elementary particle, $p_\mu = i \frac{\partial}{\partial x^\mu}$ are the components of the four-momentum operators defined in natural units ($c = 1$, $\hbar = 1$). Tensor indices are denoted with Greek letters, $\mu = 0, 1, 2, 3$ and we shall always sum over repeated indices. The γ 's are 4×4 anticommuting Dirac matrices: $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} I$ where g stands for the Minkowski space-time metric tensor: $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and I is a unit matrix. The wave function is a bispinor, i.e. consists of two two-component spinors ξ, η : $\Psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$.

In the present paper we study subsolutions of the Dirac equation. The known examples of such subsolutions are the Weyl neutrino and the Majorana neutrino, discussed in the next Section. In Section 3 the Dirac equation for a massive particle is decomposed into two separate equations. Lorentz covariance of these equations is demonstrated in Section 4 where a symmetry of subsolutions is also found. Finally, possibility that such particles can be identified with weakly interacting massive particles, other than neutrinos and of non-exotic nature, is considered.

II. WEYL AND MAJORANA NEUTRINOS AS SUBSOLUTIONS OF THE DIRAC EQUATION

In the $m = 0$ case it is possible to obtain two independent equations for spinors ξ, η by application of projection operators $Q_\pm = \frac{1}{2}(1 \pm \gamma^5)$ to Eq.(1) since $\gamma^5 \stackrel{df}{=} -i\gamma^0\gamma^1\gamma^2\gamma^3$ anticommutes with $\gamma^\mu p_\mu$:

$$Q_\pm \gamma^\mu p_\mu \Psi = \gamma^\mu p_\mu (Q_\mp \Psi) = 0. \quad (2)$$

In the spinor representation of the Dirac matrices [3] we have $\gamma^5 = \text{diag}(-1, -1, 1, 1)$ and thus $Q_- \Psi = \begin{pmatrix} \xi \\ 0 \end{pmatrix}$, $Q_+ \Psi = \begin{pmatrix} 0 \\ \eta \end{pmatrix}$ and separate equations for ξ, η follow:

$$(p^0 + \vec{\sigma} \cdot \vec{p}) \eta = 0, \quad (3a)$$

$$(p^0 - \vec{\sigma} \cdot \vec{p}) \xi = 0, \quad (3b)$$

where $\vec{\sigma}$ denotes the vector built of the Pauli matrices. Equations (3) are known as the Weyl equations and are used to describe massless left-handed and right-handed neutrinos. However, since the experimentally established phenomenon of neutrino oscillations requires non-zero neutrino masses, theory of massive neutrinos, which can be based on the Dirac equation, is necessary [5, 6, 7]. Alternatively, a modification of the Dirac or Weyl equation, called the Majorana equation, is thought to apply to neutrinos. According to Majorana theory neutrino and antineutrino are identical and neutral [8]. Although the Majorana equations can be introduced without any reference to the Dirac equation they are subsolutions of the Dirac equation [7].

Indeed, demanding in (1) that $\Psi = \mathcal{C}\Psi$ where \mathcal{C} is the charge conjugation operator, $\mathcal{C}\Psi = i\gamma^2\Psi^*$, we obtain in the spinor representation $\xi = -i\sigma^2\eta^*$, $\eta = i\sigma^2\xi^*$ and the Dirac equation (1) reduces to two separate Majorana equations for two-component spinors:

$$(p^0 + \vec{\sigma} \cdot \vec{p}) \eta = -im\sigma^2\eta^*, \quad (4a)$$

$$(p^0 - \vec{\sigma} \cdot \vec{p}) \xi = +im\sigma^2\xi^*. \quad (4b)$$

It follows from the condition $\Psi = \mathcal{C}\Psi$ that Majorana particle has zero charge built-in condition. The problem whether neutrinos are described by the Dirac equation or the Majorana equations is still open [5, 6, 7].

III. MASSIVE SUBSOLUTIONS OF THE FREE DIRAC EQUATION

A. The spinor representation

The free Dirac equation (1) in the spinor representation of γ matrices reads:

$$\left. \begin{aligned} (p^0 + p^3) \eta_1 + (p^1 - ip^2) \eta_2 &= m\xi^1 \\ (p^1 + ip^2) \eta_1 + (p^0 - p^3) \eta_2 &= m\xi^2 \\ (p^0 - p^3) \xi^1 + (-p^1 + ip^2) \xi^2 &= m\eta_1 \\ (-p^1 - ip^2) \xi^1 + (p^0 + p^3) \xi^2 &= m\eta_2 \end{aligned} \right\}, \quad (5)$$

with $\Psi = (\xi^1, \xi^2, \eta_1, \eta_2)^T$ [3] (see also [9] for full exposition of spinor formalism).

For $m \neq 0$ we can define new quantities:

$$(p^0 + p^3) \eta_1 = m\xi_{(1)}^1, \quad (p^1 - ip^2) \eta_2 = m\xi_{(2)}^1, \quad (6a)$$

$$(p^1 + ip^2) \eta_1 = m\xi_{(1)}^2, \quad (p^0 - p^3) \eta_2 = m\xi_{(2)}^2, \quad (6b)$$

where we have:

$$\xi_{(1)}^1 + \xi_{(2)}^1 = \xi^1, \quad (7a)$$

$$\xi_{(1)}^2 + \xi_{(2)}^2 = \xi^2. \quad (7b)$$

In spinor notation $\xi_{(1)}^1 = \psi_1^{1\dot{1}}$, $\xi_{(2)}^1 = \psi_2^{1\dot{2}}$, $\xi_{(1)}^2 = \psi_1^{2\dot{1}}$, $\xi_{(2)}^2 = \psi_2^{2\dot{2}}$.

Equations (5) can be now written as:

$$\left. \begin{aligned} (p^0 + p^3) \eta_1 &= m\xi_{(1)}^1 \\ (p^1 - ip^2) \eta_2 &= m\xi_{(2)}^1 \\ (p^1 + ip^2) \eta_1 &= m\xi_{(1)}^2 \\ (p^0 - p^3) \eta_2 &= m\xi_{(2)}^2 \end{aligned} \right\}. \quad (8)$$

$$\left. \begin{aligned} (p^0 - p^3) (\xi_{(1)}^1 + \xi_{(2)}^1) + (-p^1 + ip^2) (\xi_{(1)}^2 + \xi_{(2)}^2) &= m\eta_1 \\ (-p^1 - ip^2) (\xi_{(1)}^1 + \xi_{(2)}^1) + (p^0 + p^3) (\xi_{(1)}^2 + \xi_{(2)}^2) &= m\eta_2 \end{aligned} \right\}$$

It follows from Eqs.(6) that the following identities hold:

$$(p^1 + ip^2) \xi_{(1)}^1 = (p^0 + p^3) \xi_{(1)}^2, \quad (9a)$$

$$(p^0 - p^3) \xi_{(2)}^1 = (p^1 - ip^2) \xi_{(2)}^2. \quad (9b)$$

Taking into account the identities (9) we can finally write equations (8) as a system of the following two equations:

$$\left. \begin{aligned} (p^0 + p^3) \eta_i &= m \xi_{(1)}^1 \\ (p^1 + ip^2) \eta_i &= m \xi_{(1)}^2 \\ (p^0 - p^3) \xi_{(1)}^1 + (-p^1 + ip^2) \xi_{(1)}^2 &= m \eta_i \end{aligned} \right\}, \quad (10)$$

$$\left. \begin{aligned} (p^1 - ip^2) \eta_2 &= m \xi_{(2)}^1 \\ (p^0 - p^3) \eta_2 &= m \xi_{(2)}^2 \\ (-p^1 - ip^2) \xi_{(2)}^1 + (p^0 + p^3) \xi_{(2)}^2 &= m \eta_2 \end{aligned} \right\}. \quad (11)$$

Due to the identities (9) equations (10), (11) can be cast into form:

$$\left. \begin{aligned} (p^0 + p^3) \eta_i &= m \xi_{(1)}^1 \\ (p^1 + ip^2) \eta_i &= m \xi_{(1)}^2 \\ (p^0 - p^3) \xi_{(1)}^1 + (-p^1 + ip^2) \xi_{(1)}^2 &= m \eta_i \\ (p^0 + p^3) \xi_{(1)}^2 - (p^1 + ip^2) \xi_{(1)}^1 &= 0 \end{aligned} \right\}, \quad (12)$$

$$\left. \begin{aligned} (p^1 - ip^2) \eta_2 &= m \xi_{(2)}^1 \\ (p^0 - p^3) \eta_2 &= m \xi_{(2)}^2 \\ (p^0 - p^3) \xi_{(2)}^1 + (-p^1 + ip^2) \xi_{(2)}^2 &= 0 \\ (-p^1 - ip^2) \xi_{(2)}^1 + (p^0 + p^3) \xi_{(2)}^2 &= m \eta_2 \end{aligned} \right\}. \quad (13)$$

B. Arbitrary representation

Let us define the following set of mutually commuting projection operators:

$$P_1 = \frac{1}{4} (3 - \gamma^5 - \gamma^0 \gamma^3 + i \gamma^1 \gamma^2), \quad (14a)$$

$$P_2 = \frac{1}{4} (3 - \gamma^5 + \gamma^0 \gamma^3 - i \gamma^1 \gamma^2), \quad (14b)$$

$$P_3 = \frac{1}{4} (3 + \gamma^5 + \gamma^0 \gamma^3 + i \gamma^1 \gamma^2), \quad (14c)$$

$$P_4 = \frac{1}{4} (3 + \gamma^5 - \gamma^0 \gamma^3 - i \gamma^1 \gamma^2), \quad (14d)$$

where $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, introduced by Corson [9] (note that in [9] operators $\epsilon_i = 1 - P_i$ were defined and another convention for the γ matrices, $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta^{\mu\nu}I$, was used). In the spinor representation of γ matrices we have $P_1 = \text{diag}(1, 1, 1, 0)$, $P_2 = \text{diag}(1, 1, 0, 1)$.

It is now easily checked that for $m \neq 0$ Eqs.(12), (13) can be written in the spinor representation of the Dirac matrices as

$$\gamma^\mu p_\mu P_1 \Psi_{(1)} = m P_1 \Psi_{(1)}, \quad (15a)$$

$$\gamma^\mu p_\mu P_2 \Psi_{(2)} = m P_2 \Psi_{(2)}, \quad (15b)$$

respectively, where

$$\Psi_{(1)} = \begin{pmatrix} \xi_{(1)}^1 \\ \xi_{(1)}^2 \\ \eta_1 \\ \eta_2 \end{pmatrix}, \quad \Psi_{(2)} = \begin{pmatrix} \xi_{(2)}^1 \\ \xi_{(2)}^2 \\ \eta_1 \\ \eta_2 \end{pmatrix}, \quad P_1 \Psi_{(1)} + P_2 \Psi_{(2)} = \Psi. \quad (16)$$

It is important that Eqs.(15a), (15b) are valid for arbitrary representation of the Dirac matrices since the projection operators (14) are defined in arbitrary representation.

Equations (10), (11) and identities (9) can be also given representation independent form. Indeed, from (15a), (15b) the equations

$$P_i \gamma^\mu p_\mu P_i \Psi = m P_i \Psi, \quad (i = 1, 2) \quad (17a)$$

$$(1 - P_i) \gamma^\mu p_\mu P_i \Psi = 0, \quad (i = 1, 2) \quad (17b)$$

which are equivalent to Eqs.(10), (11) and (9) in the spinor representation, follow, respectively. Let us note that the identities (17b) are the direct consequence of Eqs.(15) for arbitrary representation of the Dirac matrices.

IV. LORENTZ COVARIANCE AND SYMMETRIES OF SUBSOLUTIONS

In what follows we shall use the approach of Bjorken and Drell [2] to study covariance of Eqs.(15). The postulate of Lorentz covariance means that the two equations below are equivalent:

$$\left(i \gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) P_k \Psi(x) = 0, \quad (18)$$

$$\left(i \tilde{\gamma}^\mu \frac{\partial}{\partial x^{\mu'}} - m \right) \tilde{P}'_k \Psi'(x') = 0, \quad (19a)$$

where $k = 1, 2$. Since all representations of γ matrices are unitarily equivalent [10] we may write Eq.(19a) as:

$$\left(i \gamma^\mu \frac{\partial}{\partial x^{\mu'}} - m \right) P'_k \Psi'(x') = 0. \quad (19b)$$

We start with transformation of the term $P_k \Psi(x)$ in Eq.(18):

$$\begin{aligned} P_k \Psi(x) &= P_k S^{-1}(a) \Psi'(x') = S^{-1}(a) [S(a) P_k S^{-1}(a)] \Psi'(x') = \\ &= S^{-1}(a) P'_k \Psi'(x'), \end{aligned} \quad (20)$$

where $S(a)$ is a similarity matrix depending on some set of parameters a and:

$$P'_k = S(a) P_k S^{-1}(a). \quad (21)$$

Substituting (20) into Eq.(18) we get:

$$S(a) \left(i \gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) S^{-1}(a) P'_k \Psi'(x') = 0, \quad (22)$$

$$S(a) \left(i \gamma^\mu a^\nu_\mu \frac{\partial}{\partial x^{\nu'}} - m \right) S^{-1}(a) P'_k \Psi'(x') = 0, \quad (23)$$

$$\left(i S(a) \gamma^\mu S^{-1}(a) a^\nu_\mu \frac{\partial}{\partial x^{\nu'}} - m \right) P'_k \Psi'(x') = 0. \quad (24)$$

We now demand that:

$$S(a) \gamma^\mu S^{-1}(a) a_\mu^\nu = \gamma^\nu. \quad (25)$$

Equations (25) have the following solutions:

$$S = \exp\left(-\frac{i}{4}\omega\sigma_{\mu\nu}I^{\mu\nu}\right), \quad \sigma_{\mu\nu} \stackrel{df}{=} \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \quad (26)$$

where constant matrices $I^{\mu\nu}$ define a Lorentz transformation [2].

Substituting Eq.(25) into (24) we obtain Eq.(19b) and this completes the proof that Eqs.(18), (22) are equivalent. It thus follows that Eqs.(15) are covariant.

There is a symmetry connecting equations (15a), (15b). To show this let us notice that it is possible to carry out Lorentz transformation to cast equations (15a), (15b) into the following forms:

$$(\gamma^0 p_0 - \gamma^1 p_1 - m) P_1 \Psi_{(1)}(x) = 0, \quad (27)$$

$$(\gamma^0 p_0 - \gamma^1 p_1 - m) P_2 \Psi_{(2)}(x) = 0, \quad (28)$$

respectively, since the Lorentz transformations in the $x^0 x^3$ and $x^1 x^2$ planes, are obtained from (26) as $S_{03} = e^{-\frac{i}{2}\omega\sigma_{03}}$ ($I^{30} = -I^{03} = -1$), $S_{12} = e^{\frac{i}{2}\omega\sigma_{12}}$ ($I^{12} = -I^{21} = -1$), respectively [2], commute with P_i 's. Now it turns out that the unitary matrix V :

$$V = i\gamma^2\gamma^3, \quad (29)$$

commutes with γ^0, γ^1 and transforms $P_1 \rightleftharpoons P_2$ and thus transforms Eqs. (27), (28) one into another in this special frame of reference.

V. DISCUSSION

It was demonstrated that, besides Weyl and Majorana neutrinos, there are also other subsolutions of the Dirac equation, c.f. Eqs.(15). These new subsolutions are covariant and are valid for arbitrary representation of the Dirac matrices.

Derivation of Eqs. (12), (13) in the spinor representation of the Dirac matrices was based on the identities (9). Analogous identities (17b) are fulfilled in the case of arbitrary representation of the Dirac matrices. These identities are a consequence of commutation of components of four-momentum operator in the free case. It follows that in the interacting case, these equations cannot be derived from the corresponding Dirac equation. Moreover, the mass in Eqs. (15) cannot be zero and these equations are not chiral. It thus follows that equations (15) describe some massive, spin- $\frac{1}{2}$, non-interacting and non-chiral particles. It seems natural to identify these particles with weakly interacting massive particles (WIMPs), other than neutrinos, which are candidates for dark matter, the existence of which was recently confirmed [11]. A strong candidate for WIMP dark matter is exotic particles, possibly the neutralino [12, 13]. On the other hand, theory based on Eqs.(15) does not invoke exotic matter.

Let us note finally that equations (12), (13) and their 3×3 analogues were also found in the Kemmer-Duffin-Petiau theory of spin 0 and 1 mesons [14].

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- [1] P.A.M. Dirac, The Quantum Theory of the Electron, Proc. Roy. Soc. London, A **117** (1928) 610-624; A **118** (1928) 351-361.
 - [2] J.D. Bjorken, S.D. Drell, *Relativistic Quantum Mechanics*, McGraw-Hill, Inc., 1964.
 - [3] V.B. Berestetskii, E.M. Lifshits, L.P. Pitaevskii, *Relativistic quantum theory* (Vol. 4 of the Course on Theoretical Physics), Pergamon, Oxford, 1974, part I.
 - [4] B. Thaller, *The Dirac Equation*, Springer-Verlag, Berlin, Heidelberg, New York, 1992.
 - [5] D.H. Perkins, *Introduction to high energy physics*, fourth edition, the Press Syndicate of the University of Cambridge, Cambridge, United Kingdom, 2000.
 - [6] M. Fukugita, T. Yanagida, *Physics of Neutrinos and Applications to Astrophysics*, Springer-Verlag, Berlin, Heidelberg, New York, 2003.
 - [7] M. Zralek, On the possibilities of distinguishing Dirac from Majorana neutrinos, Acta Phys. Polon. B **28** (1997) 2225 - 2257.
 - [8] E. Majorana, Teoria simmetrica dell'elettrone e del positrone, Nuovo Cimento, **14** (1937) 171 - 184.
 - [9] E.M. Corson, *Introduction to Tensors, Spinors, and Relativistic Wave-Equations (Relation Structure)*, Blackie & Son Limited, London and Glasgow, 1953.
 - [10] R.H. Good, Jr., Properties of the Dirac Matrices, Rev. Mod. Phys. **27** (1995) 187-211.
 - [11] R. Massey et al., Dark matter maps reveal cosmic scaffolding, Nature 445 (2007) 286-290 .
 - [12] G. Berone, D. Hooper, J. Silk, Particle Dark Matter: Evidence, Candidates and Constraints, Phys. Rep. **405** (2005) 279-390.
 - [13] R.W. Schnee, Status of Direct Searches for WIMP Dark Matter, astro-ph/0612565.
 - [14] A. Okninski, Splitting the Kemmer-Duffin-Petiau Equations, math-ph/0309013.